Int. J. Solids Structures, 1973, Vol. 9, pp. 1075 to 1085. Pergamon Press. Printed in Great Britain.

LONGITUDINAL SHEAR WAVES IN A FIBER-REINFORCED COMPOSITE

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Abstract—The propagation of time harmonic longitudinal shear waves in a composite with randomly distributed parallel fibers is studied. Assuming the composite to be statistically uniform, the phase velocity and the damping of the average waves are obtained as functions of the statistical and the mechanical parameters of the system. The theory leads to the well known Hashin and Rosen's formula for the axial shear modulus if the correlation in the positions of the fibers are ignored. The correlation terms are shown to have a significant effect on the damping property of the composite, especially at high frequencies and concentrations.

INTRODUCTION

WE CONSIDER a fiber-reinforced composite which consists of a homogeneous, isotropic medium, containing long, parallel, randomly distributed, circular fibers of identical properties. The theoretical investigation of the dynamic properties of such composites is a prerequisite to the design of composites with high strength and high damping. Although the statistical nature of the problem has long been recognized (Adams and Tsai [1]), a systematic statistical theory does not exist in the literature. Achenbach and Herrmann [2] included the matrix–fiber interaction by using an effective stiffness method, in which the displacement of the central line of the fiber is equated to the displacement of the matrix. Their predicted results gave no dispersion and attenuation of waves propagating normal to the fibers.

We present herein a statistical analysis of time harmonic shear waves propagating normal to the fibers. The problem of the propagation of multiply scattered waves (scalar or vector), from a random distribution of objects, has been extensively studied in the literature. Here we shall use a method similar to those of Waterman and Truell [3] and Fikioris and Waterman [4]. First, the problem of the scattering of a plane wave by a large number N of fibers, arbitrarily distributed in an infinite matrix, is considered. The resulting equations are then averaged, considering the positions of the fibers to be random. The averaged equations are solved by using Lax's "quasicrystalline" approximation [5], to yield the propagation characteristics of the average wave. This approximation, which is exact when the scatterers have causal distribution, is known to be good, even for high concentrations of the scatterers.

The results obtained here are valid for arbitrary concentration of the fibers and for arbitrary contrast in the elastic properties of the fibers and the matrix. The "pair correlation function" can also be of arbitrary form, provided its effect is assumed to be small. This assumption has been justified on the ground that when it is ignored, Hashin and Rosen's formula [6] is obtained for the axial shear. The wave length of the propagating waves are, however, assumed to be large compared to the fiber radius.

The particular case when the pair correlation function has an exponential form, is examined in detail. The average velocity of propagation and the damping are calculated under the assumption that the wave length is large compared to the "correlation length" as well as the fiber radius. The results are presented to the lowest order in the small quantities only. However, the method is general enough, so that higher order terms in these approximations can be obtained without much difficulty.

1. *Arbitrary configuration of* N *fibers*

We suppose the matrix to be extended to infinity and the fibers to be located within a large region S of a cross section of the material. Let μ, ρ be the rigidity and the density of the matrix and μ', ρ' those of the fibers. If the matrix is viscoelastic, as in a plastic material, μ can be considered complex (Hashin [7]). Labelling the fibers by suffixes $1, 2, \ldots N$ and taking suitable coordinate axes in a transverse plane, let the boundary of the ith fiber be denoted by C_i and the polar coordinates of its center 0_i be (r_i, θ_i). We consider the case when C_i's are all circles of equal radius a.

Suppose that a time harmonic transverse plane wave is generated at infinity propagating in a direction perpendicular to the fibers. Then, choosing the x-axis in the propagation direction, and suppressing the time factor $e^{-i\omega t}$, the total displacement w (parallel to the fibers) at a point P in the matrix can be written as

$$w = e^{ikx} + \sum_{i=1}^{N} \sum_{m=-\infty}^{\infty} A_{im} H_m(kR_i)\, e^{im\phi_i}, \tag{1}$$

where

$$k = \omega/\beta, \qquad \beta = (\mu/\rho)^{\frac{1}{2}}, \tag{2}$$

H_m is a Hankel function of the first kind and (R_i, ϕ_i) are the polar coordinates of P with 0_i as origin (Fig. 1). In equation (1) the first term corresponds to the incident wave and the second term represents the scattered waves emanating from the N fibers. The displacement w_i inside C_i can be similarly written as

$$w_i = \sum_{m=-\infty}^{\infty} B_{im} J_m(k'R_i)\, e^{im\phi_i}, \tag{3}$$

where

$$k' = \omega/\beta', \qquad \beta' = (\mu'/\rho')^{\frac{1}{2}}. \tag{4}$$

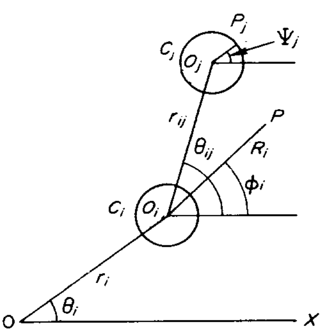

FIG. 1. The coordinate systems.

The coefficients A_{im}, B_{im} must be determined from the boundary conditions on C_j. The condition of continuity of displacement at $P_j(a, \psi_j)$ gives

$$\sum_{m=-\infty}^{\infty} B_{jm} J_m(k'a)\, e^{im\psi_j} = \left[e^{ikx} + \sum_{i=1}^{N} \sum_{m=-\infty}^{\infty} A_{im} H_m(kR_i)\, e^{im\phi_i} \right]_{P_j}.$$

Multiplying by $e^{-in\psi_j}$ and integrating from 0 to 2π, or equivalently, equating the Fourier coefficients on the two sides, we have,

$$B_{jn} J_n(k'a) = i^n J_n(ka)\, e^{ikr_j \cos\theta_j} + \sum_{i=1}^{N} \sum_{m=-\infty}^{\infty} A_{im} K_{ijmn}, \tag{5}$$

where

$$K_{ijmn} = \frac{1}{2\pi} \int_0^{2\pi} [H_m(kR_i)\, e^{im\phi_i}]_{P_j}\, e^{-in\psi_j}\, d\psi_j. \tag{6}$$

For $i = j$, we easily get

$$K_{jjmn} = H_n(ka) \delta_{mn}. \tag{7}$$

For $i \neq j$, using the addition theorem of Hankel functions (cf., Fig. 1 when P coincides with P_j)

$$[e^{im\phi_i} H_m(kR_i)]_{P_j} = e^{im\theta_{ij}} (-1)^m \sum_{s=-\infty}^{\infty} (-1)^s J_s(ka) H_{s-m}(kr_{ij})\, e^{is(\psi_j - \theta_{ij})},$$

we get

$$K_{ijmn} = J_n(ka) H_{m-n}(kr_{ij})\, e^{i(m-n)\theta_{ij}}. \tag{8}$$

Here (r_{ij}, θ_{ij}) are the polar coordinates of 0_j referred to 0_i as origin. Thus equation (5) becomes, with an obvious transformation,

$$B_{jn} J_n(k'a) = A_{jn} H_n(ka) + J_n(ka) \left[i^n\, e^{ikr_j \cos\theta_j} + \sum_{i=1}^{N}{}' \sum_{m=-\infty}^{\infty} A_{i,m+n} H_m(kr_{ij})\, e^{im\theta_{ij}} \right], \tag{9}$$

where $\sum'$ denotes the sum over all fibers except the jth.

The condition of continuity of the shear stress at P_j similarly gives,

$$\frac{\mu'}{\mu} B_{jn} \frac{\partial}{\partial a} J_n(k'a) = A_{jn} \frac{\partial}{\partial a} H_n(ka) + \frac{\partial}{\partial a} J_n(ka) \left[i^n\, e^{ikr_j \cos\theta_j} + \sum_{i=1}^{N}{}' \sum_{m=-\infty}^{\infty} A_{i,m+n} H_m(kr_{ij})\, e^{im\theta_{ij}} \right]. \tag{10}$$

From equations (9) and (10) we obtain,

$$A_{in} = iC_n F_{in}, \tag{11}$$

$$B_{in} = D_n F_{in}, \tag{12}$$

where

$$iC_n = \frac{\mu J_n(k'a)(\partial/\partial a) J_n(ka) - \mu' J_n(ka)(\partial/\partial a) J_n(k'a)}{\mu' H_n(ka)(\partial/\partial a) J_n(k'a) - \mu J_n(k'a)(\partial/\partial a) H_n(ka)}, \tag{13}$$

$$D_n = \frac{2i}{\pi a} \mu \bigg/ \left[\mu J_n(k'a) \frac{\partial}{\partial a} H_n(ka) - \mu' H_n(ka) \frac{\partial}{\partial a} J_n(k'a) \right] \tag{14}$$

and using equation (11), F_{jn} can be shown to satisfy the infinite system of linear equations

$$F_{jn} = i^n e^{ikr_j \cos\theta_j} + i \sum_{i=1}^{N}{}' \sum_{m=-\infty}^{\infty} C_{m+n} F_{i,m+n} H_m(kr_{ij}) e^{im\theta_{ij}}. \tag{15}$$

Evidently C_n and D_n are even in n.

For the case of thin fibers ($ka \ll 1$), expanding the Bessel functions and retaining the lowest order terms in ka,

$$C_0 = \frac{\pi}{4}(ka)^2\left(\frac{\rho'}{\rho}-1\right), \tag{16a}$$

$$C_1 = -\frac{\pi}{4}(ka)^2 \frac{\mu'/\mu-1}{\mu'/\mu+1}, \tag{16b}$$

$$C_m = 0, \qquad m \geq 2. \tag{16c}$$

In this case, it is evident from equation (11) that the summation in (1) consists of only three terms corresponding to $m = 0, \pm 1$, and the coefficients F_{im}, $|m| \geq 2$ are redundant. The remaining three coefficients are then given by three of the equations (15).

2. *Random distribution of fibers*

If the positions of the fibers are considered to be random in the above formulation, (R_i, ϕ_i), (r_{ij}, θ_{ij}), w, w_i and F_{im} are all random functions of some or all of $0_i(r_i, \theta_i)$. Denoting the position vector of this point by $\mathbf{r}_i$, let the probability density of the random variable $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ be $p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$. Then, due to the indistinguishability of the fibers, it is symmetric in its arguments. Furthermore, we have

$$\begin{aligned} p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) &= p(\mathbf{r}_i)p(\mathbf{r}_1, \mathbf{r}_2, \ldots, ', \ldots, \mathbf{r}_N|\mathbf{r}_i) \\ &= p(\mathbf{r}_i)p(\mathbf{r}_j|\mathbf{r}_i)p(\mathbf{r}_1, \mathbf{r}_2, \ldots, ', \ldots, ', \ldots, \mathbf{r}_N|\mathbf{r}_j, \mathbf{r}_i). \end{aligned} \tag{17}$$

Due to indistinguishability again, we have

$$p(\mathbf{r}_i) = p(\mathbf{r}_1) \quad \text{and} \quad p(\mathbf{r}_j|\mathbf{r}_i) = p(\mathbf{r}_2|\mathbf{r}_1) \quad \text{for all } i, j\ (i \neq j).$$

For a uniform composite, the positions of a single fiber are equally probable within S. Hence recalling that the probability densities are normalized and neglecting the contributions due to the finite cross section of a single fiber, which vanishes as $S \to \infty$,

$$\begin{aligned} p(\mathbf{r}_1) &= \frac{1}{S}, \qquad \mathbf{r}_1 \in S \\ &= 0, \qquad \mathbf{r}_1 \notin S. \end{aligned} \tag{18}$$

In this case, the distribution of fibers around the fiber at $0_1(\mathbf{r}_1)$, well within S, will, on the average, be circularly symmetrical. Moreover, if thin circular rings of equal width are drawn around 0_1, the number of fibers within each ring will increase, as the radius of the ring increases. The initial rate of increase should be greater for smaller concentration $c(=\pi a^2 N/S)$ of the fibers. Hence, $p(\mathbf{r}_2|\mathbf{r}_1)$ should be an increasing function of r_{12} alone, with greater initial gradient for sparser concentration. Thus we can write,

$$p(\mathbf{r}_2|\mathbf{r}_1) = \frac{1}{S}[1-f(r_{12})], \qquad \mathbf{r}_2 \in S,$$
$$= 0, \qquad \mathbf{r}_2 \notin S, \tag{19}$$

where the "pair correlation function" $f(r_{12}) \leq 1$ is a decreasing function of r_{12}. The normalization condition gives, in the limit as $S \to \infty$,

$$\lim_{R\to\infty} \frac{1}{R^2}\int_0^R f(r_{12})r_{12}\,\mathrm{d}r_{12} = 0. \tag{20}$$

Two additional restrictions are obtained from the conditions of the impossibility of interpenetration of the fibers and of their independence when they are infinitely apart. These are

$$f(r_{12}) = 1, \qquad r_{12} < 2a,$$
$$= 0, \qquad r_{12} \to \infty. \tag{21}$$

A function satisfying these conditions is

$$f(r_{12}) = 1, \qquad r_{12} < 2a,$$
$$= A\,\mathrm{e}^{-r_{12}/L}, \qquad r_{12} \geq 2a, \tag{22}$$

$0 < A \leq \mathrm{e}^{2a/L}$, and is schematically shown in Fig. 2. The correlation length $L > 0$ is such that the smaller its value the steeper the exponential part of the curve. This should take place for sparser concentrations. Thus we should have

$$L = L(c), \tag{23}$$

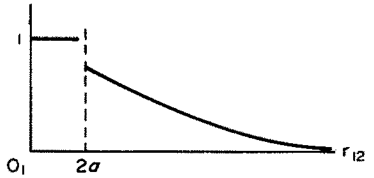

FIG. 2. The exponential pair correlation function.

where $L(c)$ is an increasing function of c. When $L \to 0$, $p(r_2|r_1)$ becomes independent of r_1, corresponding to the case of zero concentration. Hence we should have

$$\lim_{c\to 0} L(c) = 0. \tag{24}$$

Since the possible spread of the number of fibers around 0_1 is 0 to 6—which is insignificant in a large sample, the coefficient A should be nearer to its upper limit.

In the following we shall assume that the conditional probability $p(r_2|r_1)$, in general, satisfies equations (19–21) and is given by equations (22–24) as an example. Since S will be made infinite, it is assumed to be valid throughout S.

3. *Average field in an infinite composite*

Following Waterman and Truell [3], we introduce the symbol

$$\alpha(\mathbf{r}, \mathbf{r}_i) = 0, \qquad \mathbf{r} \text{ within } C_i,$$
$$= 1, \qquad \mathbf{r} \text{ outside } C_i.$$

Then, since there is no interpenetration of fibers, equations (1) and (3) can be combined to give the total displacement in the form

$$\begin{aligned} W &= \left[1-\sum_{i=1}^{N}\{1-\alpha(\mathbf{r},\mathbf{r}_i)\}\right]w+\sum_{i=1}^{N}\{1-\alpha(\mathbf{r},\mathbf{r}_i)\}w_i \\ &= \left[1-\sum_{i=1}^{N}\{1-\alpha(\mathbf{r},\mathbf{r}_i)\}\right]e^{ikx}+i\sum_{i=1}^{N}\sum_{m=-\infty}^{\infty}\alpha(\mathbf{r},\mathbf{r}_i)C_mF_{im}H_m(kR_i)\,e^{im\phi_i} \\ &\quad -\sum_{j=1}^{N}{\sum_{i=1}^{N}}'\sum_{m=-\infty}^{\infty}\{1-\alpha(\mathbf{r},\mathbf{r}_j)\}C_mF_{im}H_m(kR_i)\,e^{im\phi_i} \\ &\quad +\sum_{i=1}^{N}\sum_{m=-\infty}^{\infty}\{1-\alpha(\mathbf{r},\mathbf{r}_i)\}D_mF_{im}J_m(k'R_i)\,e^{im\phi_i}. \end{aligned} \tag{26}$$

Taking expectation values and using equations (17–19) the average field within the composite is given by

$$\begin{aligned} \langle W\rangle &= (1-c)\,e^{ikx}+n_0 i\sum_{m=-\infty}^{\infty}C_m\Bigg[\int_{|\mathbf{r}_1-\mathbf{r}|>a}\langle F_{1m}\rangle_1 H_m(kR_1)\,e^{im\phi_1}\,d\mathbf{r}_1 \\ &\quad -n_0\left(1-\frac{1}{N}\right)\int_{|\mathbf{r}_1-\mathbf{r}|\le a}d\mathbf{r}_1\int_{|\mathbf{r}_2-\mathbf{r}_1|>2a}\{1-f(\mathbf{r}_{12})\}\langle F_{2m}\rangle_{12}H_m(kR_2)\,e^{im\phi_2}\,d\mathbf{r}_2\Bigg] \\ &\quad +n_0\sum_{m=-\infty}^{\infty}D_m\int_{|\mathbf{r}_1-\mathbf{r}|\le a}\langle F_{1m}\rangle_1 J_m(k'R_1)\,e^{im\phi_1}\,d\mathbf{r}_1, \end{aligned} \tag{27}$$

where $n_0 = N/S$ is the number of fibers per unit area and $\langle F_{1m}\rangle_1$, $\langle F_{1m}\rangle_{12}$ are the conditional expectations when fibers 0_1 or 0_1 and 0_2 both are held fixed.

To determine $\langle F_{1m}\rangle_1$, we put $j = 1$ in equation (15) and take the conditional expectation to obtain

$$\begin{aligned} \langle F_{1n}\rangle_1 &= i^n\,e^{ikr_1\cos\theta_1}+in_0\left(1-\frac{1}{N}\right)\sum_{m=-\infty}^{\infty}C_{m+n}\int_{r_1,r_2\in S}\{1-f(r_{12})\} \\ &\quad \times\langle F_{2,m+n}\rangle_{12}H_m(kr_{12})\,e^{im\theta_{21}}\,d\mathbf{r}_2. \end{aligned} \tag{28}$$

This involves the unknown conditional expectation when two fibers are held fixed. If, in a similar manner, we take the conditional expectation of equation (15) with two fibers held fixed, the resulting equation will contain the conditional expectation with three fibers held fixed, and so on. We break this hierarchy, by making Lax's "quasicrystalline" approximation [5],

$$\langle F_{im}\rangle_{ij} = \langle F_{im}\rangle_i, \qquad i \ne j, \tag{29}$$

which is known to be a good approximation even for dense systems. We further use, following Lax [5], the so-called "extinction theorem" when S and N become infinitely large. According to this theorem, the incident wave is extinguished on entering the composite, so that the corresponding terms in equations (27) and (28) can be dropped. Thus, these equations reduce to,

$$\langle W\rangle = in_0 \sum_{m=-\infty}^{\infty} C_m\left[\int_{|\mathbf{r}_1-\mathbf{r}|>a} \langle F_{1m}\rangle_1 H_m(kR_1)\,\mathrm{e}^{im\phi_1}\,\mathrm{d}\mathbf{r}_1\right.$$
$$\left. -n_0\int_{|\mathbf{r}_1-\mathbf{r}|\le a}\mathrm{d}r_1\int_{|\mathbf{r}_2-\mathbf{r}_1|>2a}\{1-f(r_{12})\}\langle F_{2m}\rangle_2 H_m(kR_2)\,\mathrm{e}^{im\phi_2}\,\mathrm{d}\mathbf{r}_2\right]$$
$$+n_0\sum_{m=-\infty}^{\infty} D_m\int_{|\mathbf{r}_1-\mathbf{r}|\le a}\langle F_{1m}\rangle_1 J_m(k'R_1)\,\mathrm{e}^{im\phi_1}\,\mathrm{d}\mathbf{r}_1, \tag{30}$$

$$\langle F_{1m}\rangle_1 = in_0\sum_{n=-\infty}^{\infty} C_{m+n}\int_{|\mathbf{r}_2-\mathbf{r}_1|>2a}\{1-f(r_{12})\}\langle F_{2,m+n}\rangle_2 H_n(kr_{12})\,\mathrm{e}^{in\theta_{21}}\,\mathrm{d}\mathbf{r}_2. \tag{31}$$

Assuming the existence of an average plane wave, we try, for equation (31), the solution

$$\langle F_{im}\rangle_i = i^m F_m\,\mathrm{e}^{iKx_i}, \qquad x_i = r_i\cos\theta_i, \tag{32}$$

where F_m and K are constants. The first integral appearing in equation (31) is

$$\int_{|\mathbf{r}_2-\mathbf{r}_1|>2a}\mathrm{e}^{iKx_2}H_n(kr_{12})\,\mathrm{e}^{in\theta_{21}}\,\mathrm{d}\mathbf{r}_2$$
$$=\frac{1}{k^2-K^2}\int_{|\mathbf{r}_2-\mathbf{r}_1|>2a}[\nabla^2(\mathrm{e}^{iKx_2})H_n(kr_{12})\,\mathrm{e}^{in\theta_{21}}-\mathrm{e}^{iKx_2}\nabla^2\{H_n(kr_{12})\,\mathrm{e}^{in\theta_{21}}\}]\,\mathrm{d}\mathbf{r}_2$$
$$=\mathrm{e}^{iKx_1}\frac{2\pi a i^{-n}}{k^2-K^2}\left[J_n(2Ka)\frac{\partial}{\partial a}H_n(2ka)-H_n(2ka)\frac{\partial}{\partial a}J_n(2Ka)\right],$$

where the Green's theorem and the plane wave expansion

$$\mathrm{e}^{iKx_2}=\mathrm{e}^{iKx_1}\sum_{s=-\infty}^{\infty} i^s(-1)^s J_s(Kr_{12})\,\mathrm{e}^{is\theta_{21}},$$

have been used. The second integral in equation (31) can also be simplified by using the above expansion and equation (31) reduces to the system of equations

$$F_m = 2\pi n_0 i\sum_{n=-\infty}^{\infty} C_{m+n}F_{m+n}\left[\frac{a}{k^2-K^2}\left\{J_n(2Ka)\frac{\partial}{\partial a}H_n(2ka)-H_n(2ka)\frac{\partial}{\partial a}J_n(2Ka)\right\}\right.$$
$$\left.-\int_{2a}^{\infty} f(r_{12})J_n(Kr_{12})H_n(kr_{12})r_{12}\,\mathrm{d}r_{12}\right]. \tag{33}$$

Elimination of F_m from the above equations yields a determinantal equation of infinite order for K. It can be shown that K is the (complex) wave number of the average waves in the composite. For, evaluating the integrals in equation (30) by the methods indicated above, we get, $\langle W\rangle = \langle W\rangle_0 e^{iKx}$, where $\langle W\rangle_0$ is a constant given by

$$\langle W\rangle_0 = 2\pi a n_0 \sum_{m=-\infty}^{\infty} F_m\Bigg[\frac{iC_m}{k^2-K^2}\left\{J_m(Ka)\frac{\partial}{\partial a}H_m(ka) - H_m(ka)\frac{\partial}{\partial a}J_m(Ka)\right.$$
$$+2\pi n_0 \sum_{p=-\infty}^{\infty}\left[J_p(Ka)\frac{\partial}{\partial a}J_p(ka) - J_p(ka)\frac{\partial}{\partial a}J_p(Ka)\right]\left[\frac{a}{k^2-K^2}\left\{J_{m+p}(2Ka)\frac{\partial}{\partial a}H_{m+p}(2ka)\right.\right.$$
$$\left.\left.\left. -H_{m+p}(2ka)\frac{\partial}{\partial a}J_{m+p}(2Ka)\right\} - \int_{2a}^{\infty} J_{m+p}(Kr_{12})H_{m+p}(kr_{12})f(r_{12})r_{12}\,\mathrm{d}r_{12}\right]\right\}$$
$$+\frac{D_m}{k'^2-K^2}\left\{J_m(Ka)\frac{\partial}{\partial a}J_m(k'a) - J_m(k'a)\frac{\partial}{\partial a}J_m(Ka)\right\}\Bigg]. \tag{34}$$

4. *Thin fibers*

If the fibers are very thin compared to the wave length, ka is small and to the lowest order of approximation equation (33) is equivalent to,

$$F_m = -2\pi n_0 \sum_{n=-\infty}^{\infty} C_{m+n}F_{m+n}\left[\frac{2}{\pi}\left(\frac{K}{k}\right)^{|n|}\frac{1}{k^2-K^2}+\frac{i}{k^2}I_n\right], \tag{35}$$

where

$$I_n = \int_0^{\infty} f\left(\frac{x}{k}\right)J_n\left(\frac{K}{k}x\right)H_n(x)x\,\mathrm{d}x \tag{36}$$

and C_m is given by equations (16). Recalling the remarks following equation (16), equation (35) should be considered as three homogeneous equations in F_0, F_1, F_{-1}. The equation for K, mentioned in the preceding section, becomes in this case

$$\left(1+c\frac{m-1}{m+1}\right)\frac{K^2}{k^2} = [1+c(d-1)]\left(1-c\frac{m-1}{m+1}\right)+\frac{i\pi}{2}c\left[\left\{\frac{K^2}{k^2}\left[\frac{m-1}{m+1}-(d-1)\left(1+c\frac{m-1}{m+1}\right)\right]\right.\right.$$
$$\left.+d-1-\frac{m-1}{m+1}-2c(d-1)\frac{m-1}{m+1}\right\}I_0+4c(d-1)\frac{m-1}{m+1}\frac{K}{k}I_1$$
$$\left.-c\frac{m-1}{m+1}\left\{-\frac{K^2}{k^2}+1+c(d-1)\right\}\right]I_2$$
$$-\frac{\pi^2}{4}c^2(d-1)\frac{m-1}{m+1}\left(\frac{K^2}{k^2}-1\right)(I_0^2+I_0I_2-I_1^2) \tag{37}$$

where $d = \rho'/\rho$, $m = \mu'/\mu$. This is not an explicit equation in K, because it also appears in the integrand of I_n. If we ignore the effect of the correlation terms, an approximate solution of equation (37) is given by

$$\frac{K_0^2}{k^2} = [1+c(d-1)]\frac{1-c(m-1)/(m+1)}{1+c(m-1)/(m+1)}. \tag{38}$$

If we define the average rigidity $\langle\mu\rangle_0$ by the relation $K_0^2 = \omega^2\langle\rho\rangle/\langle\mu\rangle_0$ where $\langle\rho\rangle = c\rho' + (1-c)\rho$ is the average density, then

$$\frac{\langle\mu\rangle_0}{\mu} = \frac{1+c(m-1)/(m+1)}{1-c(m-1)/(m+1)}. \tag{39}$$

This is the well-known formula for the longitudinal shear modulus, given by Hashin and Rosen [6]. It is known to give values close to, but slightly less than, those determined by experiments, for concentrations up to 0·6. We can thus take equation (38) as the zeroth iterate of equation (37) and obtain, to the next higher order

$$\left(1+c\frac{m-1}{m+1}\right)\frac{K^2}{k^2} = [1+c(d-1)]\left(1-c\frac{m-1}{m+1}\right)$$
$$-\frac{i\pi}{2}c^2\left[\left\{(d-1)^2\left(1-c\frac{m-1}{m+1}\right)+2[1+c(d-1)]\left(\frac{m-1}{m+1}\right)^2\left(1+c\frac{m-1}{m+1}\right)^{-1}\right\}I_0\right.$$
$$\left.-4(d-1)\frac{m-1}{m+1}\frac{K_0}{k}I_1+2[1+c(d-1)]\left(\frac{m-1}{m+1}\right)\left(1+c\frac{m-1}{m+1}\right)^{-1}I_2\right], \qquad (40)$$

with K replaced by K_0 in the integrand of I_n. Although it is not difficult to obtain higher order approximations, we shall restrict ourselves to this order only.

The value of K, as determined by equation (40) is complex, say K_1+iK_2. For the relevant root $K_1, K_2 > 0$. The real part of K_1 determines the velocity of propagation $B = \omega/K_1$ and the imaginary part determines damping, a measure of which is the specific damping capacity $\psi = 4\pi K_2/K_1$.

5. *Numerical results for exponential correlation*

The general discussion of the preceding sections will now be applied to the case when $f(r_{12})$ is given by equation (22) satisfying the conditions (23) and (24). In this case we have

$$I_n = AK^2L^2\int_0^\infty e^{-x}J_n(K_0Lx)H_n(kLx)x\,dx.$$

Assuming L to be sufficiently small compared to the wave length, we obtain by expanding the Bessel functions and retaining the lowest order terms,

$$\begin{aligned} I_0 &= \frac{2i}{\pi}Ak^2L^2\left(1+\log\frac{kL}{2}-\frac{i\pi}{2}\right),\\ I_1 &= -\frac{i}{\pi}Ak^2L^2\frac{K_0}{k},\\ I_2 &= -\frac{i}{2\pi}Ak^2L^2\left(\frac{K_0}{k}\right)^2. \end{aligned} \qquad (41)$$

Thus for an elastic matrix, due to the logarithmic term in I_0, K_1 is slightly reduced from the value K_0. In other words, B is slightly increased from the corresponding value when correlations are ignored. This indicates slight dispersion of the waves. K_2 and ψ, on the other hand, are proportional to k^2L^2 and are more susceptible to changes in kL. In order to examine how B and ψ depend on c, we consider boron fibers in an aluminum matrix. For this system $d = 2{\cdot}53/2{\cdot}72$, and $m = 25/3{\cdot}87$. For the statistical parameters, we take $A = e^{2a/L} \approx 1$, because of the reasons stated under equation (24) and tentatively assume that $L(c) = L_0c$ in equation (23). The computed results are shown in Fig. 3. Both B and ψ are seen to increase with c, the rate of the increment being greater for higher concentrations. The variation of these parameters with frequency or kL_0 is also clearly brought out, and has the features mentioned above.

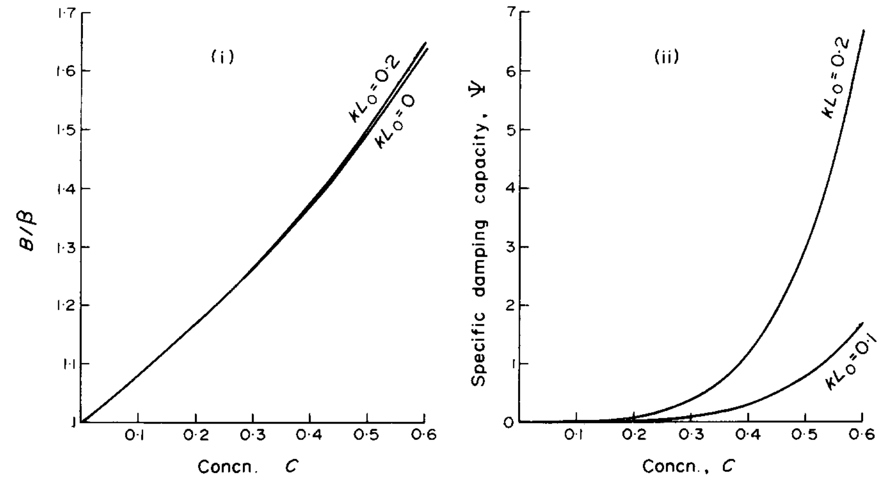

Fig. 3. Longitudinal shear wave velocity and specific damping capacity of a boron-fiber, aluminum composite.

If the matrix is a plastic, having linear visco–elastic behaviour, k and μ have to be considered as complex quantities, say k_1+ik_2 and $\mu_1-i\mu_2$ respectively, where k_2, $\mu_2 > 0$. In general μ_1 and μ_2 are functions of ω. To examine the behaviour of B and ψ in this case we consider a polyisobutylene matrix with rigid fibers for which $m \to \infty$ and $d = 2$. The dynamic properties of polyisobutylene are summarized in a paper by Toblsky and Catsiff [8], from which we can compute the values of μ_2/μ_1 for different frequencies. Assuming a

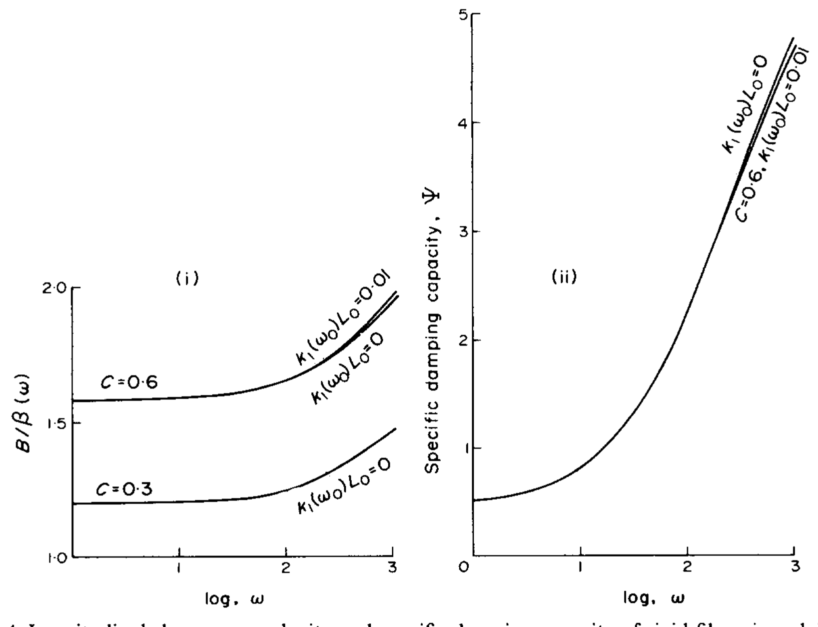

Fig. 4. Longitudinal shear wave velocity and specific damping capacity of rigid fibers in polyisobutylene matrix. In both (i) and (ii) the curves for $c = 0{\cdot}3$ and $k_1(\omega_0)L_0 = 0{\cdot}01$ deviate insignificantly from those with $k_1(\omega_0)L_0 = 0$.

base frequency of $\omega_0 = 10^{1\cdot 6437}$, we take a tentative value of L_0 given by

$$k_1(\omega_0)L_0 = 0{\cdot}01,$$

or,

$$L_0 = 0{\cdot}125 \text{ mm (approx)}.$$

The results of computation for two concentrations $c = 0{\cdot}3$ and $0{\cdot}6$ are shown in Fig. 4. Both $B/\beta(\omega)$ and ψ are found to increase with $\log_{10}\omega$. While the former increases with c the latter remains constant if correlations are ignored (cf. Hashin [7]). The correlation terms become significant for higher frequencies and concentrations. Their effect is to increase the velocity and decrease the specific damping capacity. The damping is much less affected by the correlation terms in this case due to the high damping property of polyisobutylene. But, plastics which have much less damping (like the polyester resins) will be significantly affected.

Acknowledgement—This research was supported by the National Science Foundation Grant GK 18775.

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(*Received* 26 *January* 1973)

Абстракт—Исследуется распространение периодических, гармонических, продольных волн сдвига в многослойном материале, с беспорядочно расположенными параллельными волокнами. Предполагая что многослойный материал является статистически однородным, получаются фазовая скорость и затухание средных волн в виде функций статистических и механических параметров системы. Для случая, когда пренебрегается соотношение расположения волокон, предлагаемая теория сводится к хорошо известным формулам Рашина и Розена для осевого модуля сдвига. Доказано, что члены соотношения для расположения волокон имеют значительный эффект на свойство демпфирования многоскойного материала, особенно, для случая высоких частот и концентраций.